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A NEW APPROACH FOR DEVELOPING DYNAMIC THEORIES FOR STRUCTURAL ELEMENTS

PART 2: APPLICATION TO THERMOELASTIC LAYERED COMPOSITES

YALÇIN MENGI and GÜLIN BIRLIK
The Middle East Technical University, Ankara, Turkey

and

H. D. MCNIVEN
University of California, Berkeley, CA 94720, U.S.A.

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Abstract—In this study a continuum theory is proposed which predicts the dynamic behavior of thermoelastic layered composites consisting of two alternating layers. In constructing the theory, it is noted that the governing equations for a single layer, derived in Part 1, hold in each phase of the layered composite. The theory is completed by supplementing these equations with continuity conditions and using a smoothing operation. The derivation of the continuity conditions is based on the assumption that the layers are perfectly bonded at interfaces. To assess the theory, spectra from the exact and the derived theory are compared for waves propagating in various directions of the composite. The match between the two is excellent. For waves propagating normal to layering the theory predicts both the banded and periodic structure of the spectra. The region of validity of the theory on the wave number-frequency plane can be enlargened by increasing the orders of the theory and the continuity conditions.

INTRODUCTION

Due to its importance in many fields of engineering, the dynamic behavior of layered composites has attracted the attention of many researchers. The exact treatment involves writing the exact field equations in each phase of the composite and taking into account the continuity conditions at the interfaces. Since this kind of treatment is complicated and cumbersome, researchers have attempted to develop theories in which the heterogeneous medium is replaced by a homogeneous one. Some of such theories are effective modulus theory [1, 2], effective stiffness theory [3-5], effective dispersion theory [6], mixture theory [7-9] and the theory of interacting continua [10, 11].

In this work, a new approximate theory is developed in a systematic manner using a new procedure for thermoelastic, layered composites which consist of two alternating layers. The new procedure permits us to take into account the continuity conditions at the interfaces properly and to match the exact and approximate spectra very well without using matching coefficients. The procedure starts by noting that the governing equations of a single layer established in Part 1 [12] also hold in each phase of the composite. The theory is completed by adding the continuity conditions to these equations and using a smoothing operation. The continuity conditions are derived in a unified and systematic manner by taking advantage of the fact that the face variables (which are displacements, stresses defined on the layer faces) appear as field variables in the equations of a single layer, and using the assumption that the layers are perfectly bonded. The continuity conditions thus obtained relate the face variables of two different layers.

To assess the present approximate theory, waves propagating in various directions of the layered composite are studied using various order theories and continuity conditions. As seen from the figures, the fit between the dispersion curves predicted by the exact and this approximate theory is excellent and is superior to those obtained by previous approximate theories. The banded and periodic structure of the spectra of waves propagating normal to layering is reproduced very well by the present approximate theory. The match between the exact and approximate cut-off phase velocities as well as cut-off frequencies is very good. The figures further indicate that the approximate theory is open to improvement in the sense that the region of the wave number-frequency plane over which the theory is valid can be enlarged as desired by increasing the orders of the theory and continuity conditions.

THE APPROXIMATE EQUATIONS OF THE LAYERED COMPOSITE

The layered composite under study is composed of two alternating layers perfectly bonded at their interfaces. The two different layers are indicated by the circled numbers 1 and 2 in Fig. 1. The layers 1 and 2 are assumed to be made of a linear, isotropic, thermoelastic material and have the material constants (ρ_1, μ_1, λ_1, etc.) and (ρ_2, μ_2, λ_2, etc.), and have the thicknesses $2h_1$ and $2h_2$ respectively. In the figure the pairs of the layers, each of which consists of two different phases, are numbered in increasing order $k = 0, 1, 2$, etc. In Fig. 1 two kinds of coordinate systems are shown. The first is the (x_1, x_2, x_3) global coordinate system whose (x_1, x_3) plane is parallel to the midplanes of the layers. This coordinate system is employed to designate the location of a layer by specifying the vertical distance of its midplane from the (x_1, x_3) plane. For example, $\overset{\alpha}{x}_2^{(k)}$ ($\alpha = 1, 2$) describes the position of the αth constituent of the kth pair (see Fig. 1). Here, a remark regarding the convention adopted throughout the study should be made: the Greek letters α, β, etc. are used only to distinguish the two different phases of the composite and they take the values 1 and 2. Second coordinate system is the local coordinate system (x_1, $\overset{\alpha}{x}_2$, x_3) whose (x_1, x_3) plane is chosen to coincide with the midplane of a particular layer.

The governing equations of the layered composite consist of two types of equations, namely, the field equations, valid in each layer of the composite and the continuity conditions at the interfaces. We note that the equations obtained in Part 1 [12] for a single layer hold also in each layer of the composite. Accordingly we can obtain the field equations by putting the index α in each variable appearing in the equations of the single layer. They are

equations of motion:

$$\partial_1 \overset{\alpha}{\tau}{}^n_{1i} + \partial_3 \overset{\alpha}{\tau}{}^n_{3i} - \overset{\alpha}{\bar{\tau}}{}^n_{2i} + \overset{\alpha}{R}{}_i^n + \overset{\alpha}{f}{}_i^n = \rho_\alpha \overset{\alpha}{\ddot{u}}{}_i^n \qquad (n = 0 - m), \tag{1}$$

where

$$(\overset{\alpha}{\tau}{}^n_{1i}, \overset{\alpha}{\tau}{}^n_{3i}, \overset{\alpha}{f}{}_i^n, \overset{\alpha}{u}{}_i^n) = \frac{1}{2h_\alpha}\int_{-h_\alpha}^{h_\alpha} (\overset{\alpha}{\tau}_{1i}, \overset{\alpha}{\tau}_{3i}, \overset{\alpha}{f}_i, \overset{\alpha}{u}_i)\phi_n \, d\overset{\alpha}{x}_2$$

$$\overset{\alpha}{\bar{\tau}}{}^n_{2i} = \frac{1}{2h_\alpha}\int_{-h_\alpha}^{h_\alpha} \overset{\alpha}{\tau}_{2i} \frac{d\phi_n}{d\overset{\alpha}{x}_2} d\overset{\alpha}{x}_2$$

$$\overset{\alpha}{R}{}_i^n = \frac{\phi_n(1)}{2h_\alpha} \overset{\alpha}{\overset{*}{R}}{}_i^n \tag{2}$$

$$\overset{\alpha}{\overset{*}{R}}{}_i^n = \begin{cases} \overset{\alpha}{R}{}_i^- = \overset{\alpha}{\tau}{}^+_{2i} - \overset{\alpha}{\tau}{}^-_{2i} & \text{for even } n \\ \overset{\alpha}{R}{}_i^+ = \overset{\alpha}{\tau}{}^+_{2i} + \overset{\alpha}{\tau}{}^-_{2i} & \text{for odd } n \end{cases}$$

$$\overset{\alpha}{\tau}{}^{\pm}_{2i} = \overset{\alpha}{\tau}_{2i}\Big|_{\overset{\alpha}{x}_2 = \pm h_\alpha}$$

constitutive equations:

$$\overset{\alpha}{\tau}{}^n_{11} = (2\mu_\alpha + \lambda_\alpha)\partial_1 \overset{\alpha}{u}{}_1^n + \lambda_\alpha \partial_3 \overset{\alpha}{u}{}_3^n + \lambda_\alpha(\overset{\alpha}{S}{}_2^n - \overset{\alpha}{\bar{u}}{}_2^n) - \overset{\alpha}{\beta}\overset{\alpha}{\theta}{}^n$$

$$\overset{\alpha}{\tau}{}^n_{33} = \lambda_\alpha \partial_1 \overset{\alpha}{u}{}_1^n + (2\mu_\alpha + \lambda_\alpha)\partial_3 \overset{\alpha}{u}{}_3^n + \lambda_\alpha(\overset{\alpha}{S}{}_2^n - \overset{\alpha}{\bar{u}}{}_2^n) - \overset{\alpha}{\beta}\overset{\alpha}{\theta}{}^n \tag{3}$$

$$\overset{\alpha}{\tau}{}^n_{12} = \mu_\alpha(\partial_1 \overset{\alpha}{u}{}_2^n + \overset{\alpha}{S}{}_1^n - \overset{\alpha}{\bar{u}}{}_1^n)$$

$$\overset{\alpha}{\tau}{}^{n}_{32} = \mu_\alpha(\partial_3 \overset{\alpha}{u}{}_2^n + \overset{\alpha}{S}{}_3^n - \overset{\alpha}{\bar{u}}{}_3^n)$$

$$\overset{\alpha}{\tau}{}^{n}_{13} = \overset{\alpha}{\tau}{}^{n}_{31} = \mu_\alpha(\partial_3 \overset{\alpha}{u}{}_1^n + \partial_1 \overset{\alpha}{u}{}_3^n) \qquad (n = 0 - m), \tag{3}$$

and

$$\overset{\alpha}{\bar{\tau}}{}^{n}_{22} = \lambda_\alpha \partial_1 \overset{\alpha}{\bar{u}}{}_1^n + \lambda_\alpha \partial_3 \overset{\alpha}{\bar{u}}{}_3^n + (2\mu_\alpha + \lambda_\alpha)(\overset{\alpha}{\bar{S}}{}_2^n - \overset{\alpha}{\bar{\bar{u}}}{}_2^n) - \overset{\alpha}{\beta}\, \overset{\alpha}{\bar{\theta}}{}^n$$

$$\overset{\alpha}{\bar{\tau}}{}^{n}_{21} = \mu_\alpha(\partial_1 \overset{\alpha}{\bar{u}}{}_2^n + \overset{\alpha}{\bar{S}}{}_1^n - \overset{\alpha}{\bar{\bar{u}}}{}_1^n) \tag{4}$$

$$\overset{\alpha}{\bar{\tau}}{}^{n}_{23} = \mu_\alpha(\partial_3 \overset{\alpha}{\bar{u}}{}_2^n + \overset{\alpha}{\bar{S}}{}_3^n - \overset{\alpha}{\bar{\bar{u}}}{}_3^n) \qquad (n = 0 - m),$$

where

$$\overset{\alpha}{\theta}{}^n = \frac{1}{2h_\alpha}\int_{-h_\alpha}^{h_\alpha} \overset{\alpha}{\theta}\phi_n \,\mathrm{d}\overset{\alpha}{x}_2$$

$$(\overset{\alpha}{\bar{u}}{}_i^n, \overset{\alpha}{\bar{\theta}}{}^n) = \frac{1}{2h_\alpha}\int_{-h_\alpha}^{h_\alpha} (\overset{\alpha}{u}_i, \overset{\alpha}{\theta})\frac{\mathrm{d}\phi_n}{\mathrm{d}\overset{\alpha}{x}_2}\,\mathrm{d}\overset{\alpha}{x}_2$$

$$\overset{\alpha}{\bar{\bar{u}}}{}_i^n = \frac{1}{2h_\alpha}\int_{-h_\alpha}^{h_\alpha} \overset{\alpha}{u}_i \frac{\mathrm{d}^2\phi_n}{\mathrm{d}\overset{\alpha}{x}_2{}^2}\,\mathrm{d}\overset{\alpha}{x}_2$$

$$\overset{\alpha}{S}{}_i^n = \frac{\phi_n(1)}{2h_\alpha}\overset{\alpha}{\overset{*}{S}}{}_i^n \tag{5}$$

$$\overset{\alpha}{\bar{S}}{}_i^n = \frac{\phi_n'(1)}{2h_\alpha^2}\overset{\alpha}{\overset{*}{\bar{S}}}{}_i^n$$

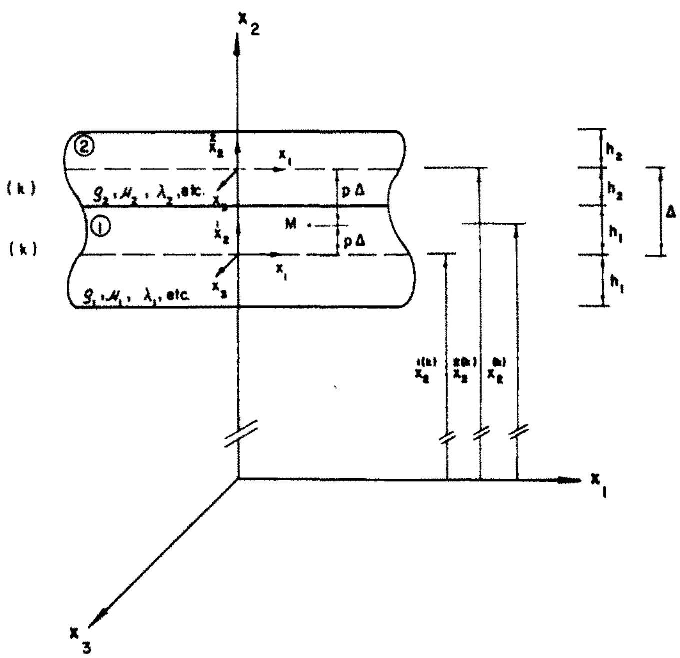

Fig. 1. Geometric description of the layered composite.

$$\overset{\alpha}{\overset{*}{S}}{}_i^n = \begin{cases} \overset{\alpha}{S}{}_i^- = \overset{\alpha}{u}{}_i^+ - \overset{\alpha}{u}{}_i^- & \text{for even } n \\ \overset{\alpha}{S}{}_i^+ = \overset{\alpha}{u}{}_i^+ + \overset{\alpha}{u}{}_i^- & \text{for odd } n \end{cases}$$

$$\overset{\alpha}{\overset{*}{\bar{S}}}{}_i^n = \begin{cases} \overset{\alpha}{S}{}_i^+ & \text{for even } n \\ \overset{\alpha}{S}{}_i^- & \text{for odd } n \end{cases}$$

$$\overset{\alpha}{u}{}_i^\pm = \overset{\alpha}{u}{}_i \Big|_{\overset{\alpha}{x}_2 = \pm h_\alpha} \tag{5}$$

$$\phi_n' = \frac{\mathrm{d}\phi_n}{\mathrm{d}\bar{x}_2}; \quad \bar{x}_2 = \frac{\overset{\alpha}{x}_2}{h_\alpha}$$

energy equation:

$$\overset{\alpha}{g}{}^n - (\partial_1 \overset{\alpha}{q}{}_1^n + \partial_3 \overset{\alpha}{q}{}_3^n + \overset{\alpha}{Q}{}^n - \overset{\alpha}{\bar{q}}{}_2^n)$$

$$= \overset{\alpha}{c}_v \overset{\alpha}{\dot{\theta}}{}^n + \overset{\alpha}{T}_0 \overset{\alpha}{\beta} (\partial_1 \overset{\alpha}{v}{}_1^n + \partial_3 \overset{\alpha}{v}{}_3^n + \overset{\alpha}{\dot{S}}{}_2^n - \overset{\alpha}{\bar{v}}{}_2^n) \quad (n = 0 - m), \tag{6}$$

where

$$\overset{\alpha}{v}{}_i^n = \overset{\alpha}{\dot{u}}{}_i^n$$

$$\overset{\alpha}{\bar{v}}{}_2^n = \overset{\alpha}{\dot{\bar{u}}}{}_2^n$$

$$(\overset{\alpha}{q}{}_1^n, \overset{\alpha}{q}{}_3^n, \overset{\alpha}{g}{}^n) = \frac{1}{2h_\alpha} \int_{-h_\alpha}^{h_\alpha} (\overset{\alpha}{q}_1, \overset{\alpha}{q}_3, \overset{\alpha}{g}) \phi_n \, \mathrm{d}\overset{\alpha}{x}_2$$

$$\overset{\alpha}{\bar{q}}{}_2^n = \frac{1}{2h_\alpha} \int_{-h_\alpha}^{h_\alpha} \overset{\alpha}{q}_2 \frac{\mathrm{d}\phi_n}{\mathrm{d}\overset{\alpha}{x}_2} \mathrm{d}\overset{\alpha}{x}_2 \tag{7}$$

$$\overset{\alpha}{Q}{}^n = \frac{\phi_n(1)}{2h_\alpha} \overset{\alpha}{\overset{*}{Q}}{}^n$$

$$\overset{\alpha}{\overset{*}{Q}}{}^n = \begin{cases} \overset{\alpha}{Q}{}^- = \overset{\alpha}{q}{}_2^+ - \overset{\alpha}{q}{}_2^- & \text{for even } n \\ \overset{\alpha}{Q}{}^+ = \overset{\alpha}{q}{}_2^+ + \overset{\alpha}{q}{}_2^- & \text{for odd } n \end{cases}$$

$$\overset{\alpha}{q}{}_2^\pm = \overset{\alpha}{q}_2 \Big|_{\overset{\alpha}{x}_2 = \pm h_\alpha}$$

Fourier's equation:

$$\overset{\alpha}{\tau}\dot{\overset{\alpha}{\bar{q}}}_1^n + \overset{\alpha}{\bar{q}}_1^n = -\overset{\alpha}{k}\partial_1\overset{\alpha}{\theta}^n$$

$$\overset{\alpha}{\tau}\dot{\overset{\alpha}{q}}_3^n + \overset{\alpha}{q}_3^n = -\overset{\alpha}{k}\partial_3\overset{\alpha}{\theta}^n \quad (n = 0 - m), \tag{8}$$

and

$$\overset{\alpha}{\tau}\dot{\overset{\alpha}{\bar{q}}}_2^n + \overset{\alpha}{\bar{q}}_2^n = -\overset{\alpha}{k}(\overset{\alpha}{\bar{\psi}}^n - \overset{\alpha}{\bar{\bar{\theta}}}^n) \quad (n = 0 - m), \tag{9}$$

where

$$\overset{\alpha}{\bar{\bar{\theta}}} = \frac{1}{2h_\alpha}\int_{-h_\alpha}^{h_\alpha} \overset{\alpha}{\theta}\frac{\mathrm{d}^2\phi_n}{\mathrm{d}\overset{\alpha}{x}_2^{\,2}}\,\mathrm{d}\overset{\alpha}{x}_2$$

$$\overset{\alpha}{\bar{\psi}}^n = \frac{\phi_n'(1)}{2h_\alpha^2}\overset{\alpha}{\overset{*}{\psi}}{}^n$$

$$\overset{\alpha}{\overset{*}{\psi}}{}^n = \begin{cases} \overset{\alpha}{\psi}{}^+ = \overset{\alpha}{\theta}{}^+ + \overset{\alpha}{\theta}{}^- & \text{for even } n \\ \overset{\alpha}{\psi}{}^- = \overset{\alpha}{\theta}{}^+ - \overset{\alpha}{\theta}{}^- & \text{for odd } n \end{cases} \tag{10}$$

$$\overset{\alpha}{\theta}{}^{\pm} = \overset{\alpha}{\theta}\Big|_{\overset{\alpha}{x}_2 = \pm h_\alpha}$$

additional equations:

$$\overset{\alpha}{S}{}_i^+ = 2\left(\sum_{k=0,2,\ldots}^{p} \gamma_k \overset{\alpha}{u}{}_i^k + \gamma^+ \overset{\alpha}{A}{}_i^+\right)$$

$$\overset{\alpha}{S}{}_i^- = 2\left(\sum_{k=1,3,\ldots}^{p'} \gamma_k \overset{\alpha}{u}{}_i^k + \gamma^- \overset{\alpha}{A}{}_i^-\right)$$

$$\overset{\alpha}{\bar{u}}{}_i^n = \begin{cases} \displaystyle\sum_{k=1,3,\ldots}^{p'} \overset{\alpha}{c}{}'_{nk}\overset{\alpha}{u}{}_i^k + \overset{\alpha}{c}{}_n'^{+}\overset{\alpha}{A}{}_i^- & \text{for even } n \\ \displaystyle\sum_{k=0,2,\ldots}^{p} \overset{\alpha}{c}{}'_{nk}\overset{\alpha}{u}{}_i^k + \overset{\alpha}{c}{}_n'^{-}\overset{\alpha}{A}{}_i^+ & \text{for odd } n \end{cases}$$

$$\overset{\alpha}{\bar{\bar{u}}}{}_i^n = \begin{cases} \displaystyle\sum_{k=0,2,\ldots}^{p} \overset{\alpha}{c}{}''_{nk}\overset{\alpha}{u}{}_i^k + \overset{\alpha}{c}{}_n''^{+}\overset{\alpha}{A}{}_i^+ & \text{for even } n \\ \displaystyle\sum_{k=1,3,\ldots}^{p'} \overset{\alpha}{c}{}''_{nk}\overset{\alpha}{u}{}_i^k + \overset{\alpha}{c}{}_n''^{-}\overset{\alpha}{A}{}_i^- & \text{for odd } n \end{cases} \tag{11}$$

and

$$\overset{\alpha}{\psi}{}^{+} = 2\Big(\sum_{k=0,2,}^{p} \gamma_k \overset{\alpha}{\theta}{}^{k} + \gamma^{+} \overset{\alpha}{B}{}^{+}\Big)$$

$$\overset{\alpha}{\psi}{}^{-} = 2\Big(\sum_{k=1,3,}^{p} \gamma_k \theta^{k} + \gamma^{-} \overset{\alpha}{B}{}^{-}\Big)$$

$$\overset{\alpha}{\bar{\theta}}{}^{n} = \begin{cases} \sum\limits_{k=1,3,..}^{p'} \overset{\alpha}{c'}_{nk} \overset{\alpha}{\theta}{}^{k} + \overset{\alpha}{c'}{}^{+}_{n} \overset{\alpha}{B}{}^{-} & \text{For even } n \\ \sum\limits_{k=0,2,..}^{p} \overset{\alpha}{c'}_{nk} \overset{\alpha}{\theta}{}^{k} + \overset{\alpha}{c'}{}^{-}_{n} \overset{\alpha}{B}{}^{+} & \text{for odd } n \end{cases} \tag{12}$$

$$\overset{\alpha}{\bar{\bar{\theta}}}{}^{n} = \begin{cases} \sum\limits_{k=0,2,..}^{p} \overset{\alpha}{c''}_{nk} \overset{\alpha}{\theta}{}^{k} + \overset{\alpha}{c''}{}^{+}_{n} \overset{\alpha}{B}{}^{+} & \text{for even } n \\ \sum\limits_{k=1,3,..}^{p'} \overset{\alpha}{c''}_{nk} \overset{\alpha}{\theta}{}^{k} + \overset{\alpha}{c''}{}^{-}_{n} \overset{\alpha}{B}{}^{-} & \text{for odd } n, \end{cases}$$

where $p = m$, $p' = m - 1$ for even m, $p = m - 1$, $p' = m$ for odd m; m: the order of the theory. In eqns (1)–(2) it is assumed that the same distribution functions ϕ_n are used for both of the phases. The dot denotes the differentiation with respect to time t and $\partial_i = (\partial/\partial x_i)$. It is assumed that the summation convention does not apply to Greek indices. The definitions of the variables and constants appearing in eqns (1)–(12) can be found from those given in Part 1 by putting the index α on each term of the equations of the single layer. When the ϕ_n are Legendre polynomials the values of γ_k, $\gamma^{\pm}$, $\overset{\alpha}{c'}_{nk}$, $\overset{\alpha}{c'}{}^{\pm}_{n}$, $\overset{\alpha}{c''}_{nk}$, $\overset{\alpha}{c''}{}^{\pm}_{n}$ in eqns (11), (12) can be obtained from Table 1 of [12] by replacing h by h_α.

Before writing the continuity conditions, it is to be noted that there are two kinds of interfaces: one follows the layer 1 and the other layer 2. As it is assumed that there is perfect bonding between the layers, the displacements u_i, the stress components τ_{2i}, the heat flux component q_2 and the temperature θ should be continuous across these interfaces, i.e. for the interface following the layer 1:

$$\overset{1}{u}{}_i^{+} = \overset{2}{u}{}_i^{-}; \quad \overset{1}{\tau}{}_{2i}^{+} = \overset{2}{\tau}{}_{2i}^{-}; \quad \overset{1}{q}{}_2^{+} = \overset{2}{q}{}_2^{-}; \quad \overset{1}{\theta}{}^{+} = \overset{2}{\theta}{}^{-} \tag{13}$$

Table 1. Properties of thornel-carbon phenolic composite

h_1	h_2	$n_1 = \frac{h_1}{\Delta}$	$n_2 = \frac{h_2}{\Delta}$	ρ_1	ρ_2
cm				$\frac{\text{dyne} - \mu\text{sec}^2}{\text{cm}^4}$	
0.0032	0.0279	0.103	0.897	1.47 $\times 10^{12}$	1.42 $\times 10^{12}$
μ_1	μ_2	λ_1	λ_2		
dyne/cm^2		dyne/cm^2			
0.756×10^{12}	0.0662×10^{12}	0.756 $\times 10^{12}$	0.114 $\times 10^{12}$		

for the interface following the layer 2:

$$\overset{2}{u}_i^+ = \overset{1}{u}_i^-;\quad \overset{2}{\tau}_{2i}^+ = \overset{1}{\tau}_{2i}^-;\ \overset{2}{q}_2^+ = \overset{1}{q}_2^-;\quad \overset{2}{\theta}^+ = \overset{1}{\theta}^-. \tag{14}$$

Using eqns $(2)_4$, $(5)_6$, $(7)_6$ and $(10)_3$ the continuity conditions, eqns (13), (14) can be expressed in terms of face variables $\overset{\alpha}{R}_i^\mp$, $\overset{\alpha}{S}_i^\mp$, $\overset{\alpha}{Q}^\mp$ and $\overset{\alpha}{\psi}^\mp$. They are for the interface following the layer 1:

$$\begin{aligned}
&\overset{2}{S}_i^+ - \overset{1}{S}_i^+ = \overset{1}{S}_i^- + \overset{2}{S}_i^-\\
&\overset{2}{R}_i^+ - \overset{1}{R}_i^+ = \overset{1}{R}_i^- + \overset{2}{R}_i^-\\
&\overset{2}{Q}^+ - \overset{1}{Q}^+ = \overset{1}{Q}^- + \overset{2}{Q}^-\\
&\overset{2}{\psi}^+ - \overset{1}{\psi}^+ = \overset{1}{\psi}^- + \overset{2}{\psi}^-
\end{aligned} \tag{15}$$

for the interface following the layer 2:

$$\begin{aligned}
&\overset{1}{S}_i^+ - \overset{2}{S}_i^+ = \overset{1}{S}_i^- + \overset{2}{S}_i^-\\
&\overset{1}{R}_i^+ - \overset{2}{R}_i^+ = \overset{1}{R}_i^- + \overset{2}{R}_i^-\\
&\overset{1}{Q}^+ - \overset{2}{Q}^+ = \overset{2}{Q}^+ = \overset{1}{Q}^- + \overset{2}{Q}^-\\
&\overset{1}{\psi}^+ - \overset{2}{\psi}^+ = \overset{1}{\psi}^- + \overset{2}{\psi}^-.
\end{aligned} \tag{16}$$

We note that the dependent variables $\overset{\alpha}{\underline{w}} = (\overset{\alpha}{u}_i^n, \overset{\alpha}{\tau}_{2i}^n, \overset{\alpha}{\tau}_{1i}^n, \overset{\alpha}{\tau}_{3i}^n, \overset{\alpha}{S}_i^\pm$, etc.) appearing in eqns (1)–(12) and eqns (15), (16) are the functions of $\overset{\alpha}{x}{}_2^{(k)}$, i.e. their values depend on the positions of the layers. Accordingly, eqns (1)–(12) and eqns (15), (16) form a discrete system of equations. In order to obtain the solution using this discrete model, one should write eqns (1)–(12) in all layers and take into account the continuity conditions, eqns (15), (16), at all interfaces. This kind of procedure involves lengthy computations and appears to be of no practical use. To simplify the analysis, in what follows we replace the discrete model by a continuous model by using a smoothing operation.

To obtain the smoothed form of the field equations, eqns (1)–(12), we first replact $\overset{\alpha}{x}{}_2^{(k)}$ in the arguments of the variables $\overset{\alpha}{\underline{w}}$ appearing in these equations by x_2. After this smoothing, $\overset{\alpha}{\underline{w}}$ is now defined for all x_2 but, it has physical meaning only at the midplanes $x_2 = \overset{\alpha}{x}{}_2^{(k)}$. The smoothing operation leaves eqns (1)–(12) unchanged because all of the field variables in these equations are defined at the midplane of the same layer, i.e. at $x_2 = \overset{\alpha}{x}{}_2^{(k)}$. In accordance with the idealization implied by the smoothing operation, it is further assumed that both types of layers exist simultaneously at every point of the continuum and accordingly, eqns (1)–(12) with $\alpha = 1$ and 2 hold at the same point x_2.

With regard to the continuity conditions, eqns (15), (16), it is to be observed that the variables appearing in these equations do not belong to the same layer. Therefore the smoothed form of the continuity conditions will change and will be found through analysis. The analysis starts by referring to Fig. 1 and writing the continuity equations for the interface which follows the layer 1, eqns (15), explicitly as

$$\overset{2}{F}{}^{+}(x_1, \overset{2}{x}{}_2^{(k)}, x_3, t) - \overset{1}{F}{}^{+}(x_1, \overset{1}{x}{}_2^{(k)}, x_3, t) = \overset{1}{F}{}^{-}(x_1, \overset{1}{x}{}_2^{(k)}, x_3, t) + \overset{2}{F}{}^{-}(x_1, \overset{2}{x}{}_2^{(k)}, x_3, t), \tag{17}$$

where $\overset{\alpha}{F}{}^{\mp}$ stands for either of the face variables $(\overset{\alpha}{S}{}_i^{\mp}, \overset{\alpha}{R}{}_i^{\mp}, \overset{\alpha}{Q}{}^{\mp}, \overset{\alpha}{\psi}{}^{\mp})$, $i = 1-3$. It must be observed that $\overset{1}{F}{}^{\mp}$ is defined at the midplane of the layer 1 (i.e. at $x_2 = \overset{1}{x}{}_2^{(k)}$) while $\overset{2}{F}{}^{\mp}$ is defined at that of the layer 2 (i.e. at $x_2 = \overset{2}{x}{}_2^{(k)}$). To apply the smoothing operations to the continuity condition, eqn (17) will be written first in a form in which all of the variables in it are defined at a single point. To this end a point M, in the interval $(x^{1}{}_2^{(k)}, \overset{2}{x}{}_2^{(k)})$ with distances $p_1\Delta$ and $p_2\Delta$ from the midplanes of the layers 1 and 2 respectively is chosen, where $\Delta = h_1 + h_2$ and p_α has the property $p_1 + p_2 = 1$ (see Fig. 1). The x_2 coordinate of this point is designated by $x_2^{(k)}$ in the figure. By taking into account the relations $\overset{1}{x}{}_2^{(k)} = x_2^{(k)} - p_1\Delta$ and $\overset{2}{x}{}_2^{(k)} = x_2^{(k)} + p_2\Delta$ the continiuty condition, eqn (17), now becomes

$$\overset{2}{F}{}^{+}(x_2^{(k)} + p_2\Delta) - \overset{1}{F}{}^{+}(x_2^{(k)} - p_1\Delta) = \overset{1}{F}{}^{-}(x_2^{(k)} - p_1\Delta) + \overset{2}{F}{}^{-}(x_2^{(k)} + p_2\Delta). \tag{18}$$

For simplicity the arguments x_1, x_3 and t of $\overset{\alpha}{F}{}^{\mp}$ are omitted in eqn (18)

The reduced form of the continuity condition for the interface which follows the layer 2 can be obtained from eqn (18) by replacing the layer index 1 by 2 and 2 by 1. It is

$$\overset{1}{F}{}^{+}(x_2^{(k)} + p_1\Delta) - \overset{2}{F}{}^{+}(x_2^{(k)} - p_2\Delta) = \overset{2}{F}{}^{-}(x_2^{(k)} - p_2\Delta) + \overset{1}{F}{}^{-}(x_2^{(k)} + p_1\Delta). \tag{19}$$

To obtain the smoothed forms of the continuity equations, it is assumed that the two types of interfaces exist simultaneously at the same point of the continuum and $x_2^{(k)}$ in eqns (18), (19) is replaced by x_2. Thus we obtain

$$\begin{aligned} &\overset{2}{F}{}^{+}(x_2 + p_2\Delta) - \overset{1}{F}{}^{+}(x_2 - p_1\Delta) = \overset{1}{F}{}^{-}(x_2 - p_1\Delta) + \overset{2}{F}{}^{-}(x_2 + p_2\Delta) \\ &\overset{1}{F}{}^{+}(x_2 + p_1\Delta) - \overset{2}{F}{}^{+}(x_2 - p_2\Delta) = \overset{2}{F}{}^{-}(x_2 - p_2\Delta) + \overset{1}{F}{}^{-}(x_2 + p_1\Delta). \end{aligned} \tag{20}$$

When the first and second of eqns (20) are added and subtracted one gets

$$\begin{aligned} &\overset{2}{F}{}^{+}(x_2 + p_2\Delta) - \overset{2}{F}{}^{+}(x_2 - p_2\Delta) + \overset{1}{F}{}^{+}(x_2 + p_1\Delta) - \overset{1}{F}{}^{+}(x_2 - p_1\Delta) \\ &= \overset{2}{F}{}^{-}(x_2 + p_2\Delta) + \overset{2}{F}{}^{-}(x_2 - p_2\Delta) + \overset{1}{F}{}^{-}(x_2 + p_1\Delta) + \overset{1}{F}{}^{-}(x_2 - p_1\Delta) \\ &= \overset{2}{F} - (x_2 + p_2\Delta) + \overset{2}{F}{}^{+}(x_2 - p_2\Delta) - (\overset{1}{F}{}^{+}(x_2 + p_1\Delta) + \overset{1}{F}{}^{+}(x_2 - p_1\Delta)) \\ &= \overset{2}{F}{}^{-}(x_2 + p_2\Delta) - \overset{2}{F}{}^{-}(x_2 - p_2\Delta) - (\overset{1}{F}{}^{-}(x_2 + p_1\Delta) - \overset{1}{F}{}^{-}(x_2 - p_1\Delta)). \end{aligned} \tag{21}$$

Expanding the terms of eqns (21) in Taylor's series about the point x_2 we finally obtain

$$\begin{aligned} &s_2\overset{2}{F}{}^{+} + s_1\overset{1}{F}{}^{+} = c_2\overset{2}{F}{}^{-} + c_1\overset{1}{F}{}^{-} \\ &c_2\overset{2}{F}{}^{+} - c_1\overset{1}{F}{}^{+} = s_2\overset{2}{F}{}^{-} - s_1\overset{1}{F}{}^{-}, \end{aligned} \tag{22}$$

where s_α and c_α are the operators defined by

$$s_\alpha = (p\alpha\Delta)\partial_2 + \frac{(p_\alpha\Delta)^3}{3!}\partial_2{}^3 + \ldots = \sinh \kappa_\alpha$$

$$c_\alpha = 1 + \frac{(p_\alpha\Delta)^2}{2!}\partial_2{}^2 + \ldots = \cosh \kappa_\alpha \tag{23}$$

$$\kappa_\alpha = (p_\alpha\Delta)\partial_2.$$

Equation (22) with $\overset{\alpha}{F}{}^{\mp} = (\overset{\alpha}{S}{}_i^{\mp}, \overset{\alpha}{R}{}_i^{\mp}, \overset{\alpha}{Q}{}^{\mp}, \overset{\alpha}{\psi}{}^{\mp})$ respectively represent the smoothed form of the continuity conditions for displacement, stress, heat flux and temperature. It must be noted that these continuity conditions have an invariant form with regard to layer indices 1 and 2, i.e. they remain unchanged when the index 1 is replaced by 2 and the index 2 by 1.

The derivation of the equations of an mth order continuum theory for a layered composite is now completed. The governing equations are composed of the field equations, eqns (1)–(12) and the continuity conditions, eqns (22). They constitute $(48(m+1)+32)$ equations for the $(48(m+1)+32)$ unknown variables $(\overset{\alpha}{u}{}_i^n, \overset{\alpha}{\bar{u}}{}_i^n, \overset{\alpha}{\bar{\bar{u}}}{}_i^n, \overset{\alpha}{\tau}{}_{1i}^n, \overset{\alpha}{\tau}{}_{3i}^n, \overset{\alpha}{\bar{\tau}}{}_{2i}^n, \overset{\alpha}{q}{}_1^n, \overset{\alpha}{q}{}_3^n, \overset{\alpha}{\bar{q}}{}_2^n, \overset{\alpha}{\theta}{}^n, \overset{\alpha}{\bar{\theta}}{}^n, \overset{\alpha}{\bar{\bar{\theta}}}{}^n, \overset{\alpha}{S}{}_i^{\mp}, \overset{\alpha}{R}{}_i^{\mp}, \overset{\alpha}{Q}{}^{\mp}, \overset{\alpha}{\psi}{}^{\mp})$. It should be emphasized that the number of equations and unknowns can be decreased through some eliminations. For example when the variables $\overset{\alpha}{\tau}{}_{1i}^n, \overset{\alpha}{\tau}{}_{3i}^n, \overset{\alpha}{\bar{\tau}}{}_{2i}^n, \overset{\alpha}{q}{}_1^n, \overset{\alpha}{q}{}_3^n, \overset{\alpha}{\bar{q}}{}_2^n, \overset{\alpha}{\bar{u}}{}_i^n, \overset{\alpha}{\bar{\bar{u}}}{}_i^n, \overset{\alpha}{\bar{\theta}}{}^n, \overset{\alpha}{\bar{\bar{\theta}}}{}^n, \overset{\alpha}{R}{}_i^{\mp}, \overset{\alpha}{Q}{}^{\mp}$ are eliminated by using eqns (3), (4), (8), (9), (11) and (12) the number of unknowns $(\overset{\alpha}{u}{}_i^n, \overset{\alpha}{\theta}{}^n, \overset{\alpha}{S}{}_i^{\pm}, \overset{\alpha}{\psi}{}^{\mp})$ and resulting governing equations reduces to $(8(m+1)+16)$. Here it must be pointed out that the mth order theory constitutes a $2(m+1)$ mode theory for principal waves, i.e. it accommodates $2(m+1)$ dispersion curves in the spectra for the waves propagating parallel and perpendicular to the layering.

Continuity conditions for the waves propagating parallel to the layering

The continuity conditions, eqns (22), are general and hold for any kind of wave propagating in an arbitrary direction. However, they assume a simpler form for waves (on the average longitudinal or transverse) propagating parallel to the layering. For such waves the field variables become independent of x_2 and consequently, in the series, eqns (23), all of the terms except the ones which do not involve the derivative of x_2 vanish. Thus the operators reduce to $s_\alpha = 0$ and $c_\alpha = 1$. With these forms of the operators the continuity conditions, eqns (22), become

$$\overset{2}{F}{}_i^- + \overset{1}{F}{}_i^- = 0; \quad \overset{2}{F}{}_i^+ - \overset{1}{F}{}_i^+ = 0 \quad F = (S, R, Q, \psi), \tag{24}$$

which are independent of the constants p_1 and p_2.

To interpret eqns (24) physically the equation with $F = S$ will be considered and the interface will be assumed to follow the layer 1. In view of the expressions defining $\overset{\alpha}{S}{}_i^{\mp}$, eqn $(5)_6$, eqns (24) then reduce to

$$\overset{1}{u}{}_i^+ = \overset{2}{u}{}_i^-; \qquad \overset{1}{u}{}_i^- = \overset{2}{u}{}_i^+. \tag{25}$$

The first of eqns (25) describes the continuity of displacements at the interface and the second implies that the actual displacement distribution is periodic with period 2Δ. Equation (24) with $F = (R, Q, \psi)$ lead to the same conclusion with regard to τ_{2i}, q_2 and θ respectively. These conditions are identical with those which are used in the exact analysis of waves propagating parallel to the layering (see [2]).

Continuity conditions for the waves propagating at a non-zero angle to layering

For such waves all of the terms in the series defining the operators, s_α, c_α, eqns (23), have to be considered and the continuity conditions become dependent on the constants p_1, p_2. By retaining certain numbers of the terms in the series various order approximate continuity conditions can be obtained. For example, when the terms including κ_α and κ_α^2 are retained in the series two different approximate forms for the continuity conditions can be obtained:

$$\Delta\partial_2(p_2\overset{2}{F}{}_i^+ + p_1\overset{1}{F}{}_i^+) = \overset{1}{F}{}_i^- + \overset{2}{F}{}_i^-$$

$$\Delta\partial_2(p_2\overset{2}{F}{}_i^- - p_1\overset{1}{F}{}_i^-) = \overset{2}{F}{}_i^+ - \overset{1}{F}{}_i^+ \qquad F = (S, R, Q, \psi), \tag{26}$$

and

$$\Delta\partial_2(p_2\overset{2}{F}{}_i^+ + p_1\overset{1}{F}{}_i^+) = (\overset{1}{F}{}_i^- + \overset{2}{F}{}_i^-) + \frac{\Delta^2}{2}\partial_2^2(p_1^2\overset{1}{F}{}_i^- + p_2^2\overset{2}{F}{}_i^-)$$

$$\Delta\partial_2(p_2\overset{2}{F}{}_i^- - p_1\overset{1}{F}{}_i^-) = (\overset{2}{F}{}_i^+ - \overset{1}{F}{}_i^+) + \frac{\Delta^2}{2}\partial_2^2(p_2^2\overset{2}{F}{}_i^+ - p_1^2\overset{1}{F}{}_i^+) \tag{27}$$

$$F = (S, R, Q, \psi).$$

In the study, eqns (26) and (27), will be referred to as first and second order continuity conditions respectively.

The numerical analysis indicates that the match between the exact and approximate results appears to be the best when $p_1 = p_2 = 0.5$ (which corresponds to taking the point M in Fig. 1 at the midpoint of the vertical distance between the midplanes of two adjacent layers) and the match improves as the order of the continuity condition is increased. This is illustrated in Fig. 2 which, in the absence of thermal effects, shows the dispersion curves for $P \to x_2$ (i.e. for P waves propagating in x_2 direction) for thornel-carbon phenolic composite whose material properties are given in Table 1 of the last section. In Fig. 2, ω and k designate the angular

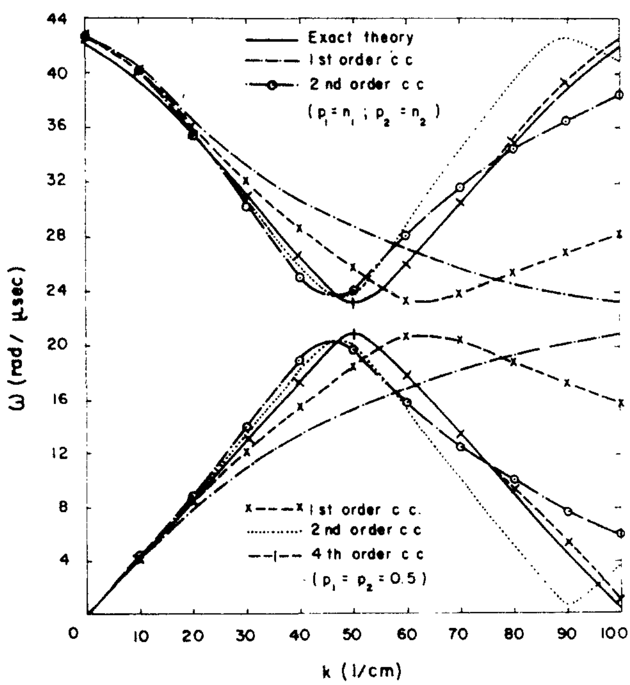

Fig. 2. Dispersion curves for $P \to x_2$ (thornel-carbon phenolic) obtained using various orders of the theory and continuity conditions and various values of p_1 and p_2.

frequency and the wave number respectively. In the figure the approximate spectra are presented for six different cases:

(i) $p_1 = p_2 = 0.5$ and first order continuity condition;
(ii) $p_1 = p_2 = 0.5$ and second order continuity condition;
(iii) $p_1 = p_2 = 0.5$ and fourth order continuity condition;
(iv) $p_1 = n_1$; $p_2 = n_2$ and first order continuity condition;
(v) $p_1 = n_1$; $p_2 = n_2$ and second order continuity condition;
(vi) $p_1 = n_1$; $p_2 = n_2$ and fourth order continuity condition;

where n_1 and n_2 are the volume fractions defined by $n_1 = (h_1/\Delta)$ and $n_2 = (h_2/\Delta)$ with the property $n_1 + n_2 = 1$. Choosing $p_1 = n_1$ and $p_2 = n_2$ corresponds to taking the point M at the interface. In all six cases the second order theory is used. The details of the procedure used to obtain the approximate spectral lines can be found in the next section. As seen from the figure, the best fit between the exact and the approximate spectra is obtained for the third case, i.e. when $p_1 = p_2 = 0.5$ and the fourth order continuity condition is used. The approximate spectrum corresponding to this case predicts not only the stopping and passing bands but also describes very well the periodic nature of the dispersion curves along the k axis.

In view of the findings established above, for the numerical analysis presented in the last section, $p_1 = p_2 = 0.5$ is chosen; and due to its simple structure and its ability to predict both the banded and periodic structure of spectra the second order continuity conditions are used.

DISPERSION RELATIONS

For the sake of brevity, we outline in this section the procedure for obtaining the approximate disperions relation only for $P \to x_1$. The dispersion relations for the other waves, namely, for $SV \to x_1$, $SH \to x_1$, $P \to x_2$, $SV \to x_2$ and the waves propagating obliquely to the layering, can be found following a similar procedure. In the analysis the thermal effects are neglected and the second ($m = 2$) and fourth ($m = 4$) order theories are considered.

For the wave $P \to x_1$ which is on the average longitudinal, the actual longitudinal displacement u_1 and transverse displacement u_2 are respectively symmetric and antisymmetric with respect to the midplanes of the layers. From the study of eqns (2) and (5), it follows that the nonzero components of generalized displacement and face variables are

$$\overset{\alpha}{u}_1{}^n,\ \overset{\alpha}{\bar{u}}_2{}^n,\ \overset{\alpha}{\bar{\bar{u}}}_1{}^n \quad \text{for even } n$$

$$\overset{\alpha}{u}_2{}^n,\ \overset{\alpha}{\bar{u}}_1{}^n,\ \overset{\alpha}{\bar{\bar{u}}}_2{}^n \quad \text{for odd } n \qquad (n = 0 - m;\ m = 2, 4)$$

and

$$\overset{\alpha}{S}_1{}^+, \overset{\alpha}{S}_2{}^-,\ \overset{\alpha}{R}_1{}^-,\ \overset{\alpha}{R}_2{}^+.$$

Accordingly, eqns (1), (3), (4), (11) and (24) reduce to
equations of motion:

$$\partial_1 \overset{\alpha}{\tau}{}^n_{11} - \overset{\alpha}{\bar{\tau}}{}^n_{21} + \overset{\alpha}{R}_1{}^n = \rho_\alpha \overset{\alpha}{\ddot{u}}_1{}^n$$

$$\partial_1 \overset{\alpha}{\tau}{}^n_{12} - \overset{\alpha}{\bar{\tau}}{}^n_{22} + \overset{\alpha}{R}_2{}^n = \rho_\alpha \overset{\alpha}{\ddot{u}}_2{}^n \ (n = 0 - m;\ m = 2, 4), \tag{28}$$

where

$$\overset{\alpha}{R}_1{}^0 = \overset{\alpha}{R}_1{}^2 = \overset{\alpha}{R}_1{}^4 = \frac{1}{2h_\alpha} \overset{\alpha}{R}_1{}^-$$

$$\overset{\alpha}{R}_2{}^1 = \overset{\alpha}{R}_2{}^3 = \frac{1}{2h_\alpha} \overset{\alpha}{R}_2{}^+$$

constitutive equations:

$$\overset{\alpha}{\tau}{}^{0}_{11} = (2\mu_\alpha + \lambda_\alpha)\partial_1 \overset{\alpha}{u}{}_1^{0} + \lambda_\alpha(\overset{\alpha}{S}{}_2^{0} - \overset{\alpha}{\bar{u}}{}_2^{0})$$

$$\overset{\alpha}{\tau}{}^{2}_{11} = (2\mu_\alpha + \lambda_\alpha)\partial_1 \overset{\alpha}{u}{}_1^{2} + \lambda_\alpha(\overset{\alpha}{S}{}_2^{2} - \overset{\alpha}{\bar{u}}{}_2^{2})$$

$$\overset{\alpha}{\tau}{}^{4}_{11} = (2\mu_\alpha + \lambda_\alpha)\partial_1 \overset{\alpha}{u}{}_1^{4} + \lambda_\alpha(\overset{\alpha}{S}{}_2^{4} - \overset{\alpha}{\bar{u}}{}_2^{4}) \tag{29}$$

$$\overset{\alpha}{\tau}{}^{1}_{12} = \mu_\alpha(\partial_1 \overset{\alpha}{u}{}_2^{1} + \overset{\alpha}{S}{}_1^{1} - \overset{\alpha}{\bar{u}}{}_1^{1})$$

$$\overset{\alpha}{\tau}{}^{3}_{12} = \mu_\alpha(\partial_1 \overset{\alpha}{u}{}_2^{3} + \overset{\alpha}{S}{}_1^{3} - \overset{\alpha}{\bar{u}}{}_1^{3}),$$

where

$$\overset{\alpha}{S}{}_2^{0} = \overset{\alpha}{S}{}_2^{2} = \overset{\alpha}{S}{}_2^{4} = \frac{1}{2h_\alpha}\overset{\alpha}{S}{}_2^{-}$$

$$\overset{\alpha}{S}{}_1^{1} = \overset{\alpha}{S}{}_1^{3} = \frac{1}{2h_\alpha}\overset{\alpha}{S}{}_1^{+}$$

and

$$\overset{\alpha}{\bar{\tau}}{}^{0}_{21} = \mu_\alpha(\partial_1 \overset{\alpha}{\bar{u}}{}_2^{0} + \overset{\alpha}{\bar{S}}{}_1^{0} - \overset{\alpha}{\bar{\bar{u}}}{}_1^{0}); \qquad \overset{\alpha}{\bar{S}}{}_1^{0} = 0$$

$$\overset{\alpha}{\bar{\tau}}{}^{2}_{21} = \mu_\alpha(\partial_1 \overset{\alpha}{\bar{u}}{}_2^{2} + \overset{\alpha}{\bar{S}}{}_1^{2} - \overset{\alpha}{\bar{\bar{u}}}{}_1^{2}); \qquad \overset{\alpha}{\bar{S}}{}_1^{2} = \frac{3}{2h_\alpha^2}\overset{\alpha}{S}{}_1^{+}$$

$$\overset{\alpha}{\bar{\tau}}{}^{4}_{21} = \mu_\alpha(\partial_1 \overset{\alpha}{\bar{u}}{}_2^{4} + \overset{\alpha}{\bar{S}}{}_1^{4} - \overset{\alpha}{\bar{\bar{u}}}{}_1^{4}); \ \overset{\alpha}{\bar{S}}{}_1^{4} = \frac{5}{h_\alpha^2}\overset{\alpha}{S}{}_1^{+} \tag{30}$$

$$\overset{\alpha}{\bar{\tau}}{}^{1}_{22} = \lambda_\alpha \partial_1 \overset{\alpha}{\bar{u}}{}_1^{1} + (2\mu_\alpha + \lambda_\alpha)(\overset{\alpha}{\bar{S}}{}_2^{1} - \overset{\alpha}{\bar{\bar{u}}}{}_2^{1}); \ \overset{\alpha}{\bar{S}}{}_2^{1} = \frac{1}{2h_\alpha^2}\overset{\alpha}{S}{}_2^{-}$$

$$\overset{\alpha}{\bar{\tau}}{}^{3}_{22} = \lambda_\alpha \partial_1 \overset{\alpha}{\bar{u}}{}_1^{3} + (2\mu_\alpha + \lambda_\alpha)(\overset{\alpha}{\bar{S}}{}_2^{3} - \overset{\alpha}{\bar{\bar{u}}}{}_2^{3}); \overset{\alpha}{\bar{S}}{}_2^{3} = \frac{3}{h_\alpha^2}\overset{\alpha}{S}{}_2^{-}$$

equations for $\overset{\alpha}{\bar{u}}{}_i^{n}$ and $\overset{\alpha}{\bar{\bar{u}}}{}_i^{n}$:

$$\overset{\alpha}{\bar{u}}{}_1^{1} = \frac{1}{h_\alpha}\overset{\alpha}{u}{}_1^{0}$$

$$\overset{\alpha}{\bar{u}}{}_1^{3} = \frac{1}{h_\alpha}\overset{\alpha}{u}{}_1^{0} + \frac{5}{h_\alpha}\overset{\alpha}{u}{}_1^{2}$$

$$\overset{\alpha}{\bar{u}}{}_2^{0} = 0 \tag{31}$$

$$\overset{\alpha}{\bar{u}}{}_2^{2} = \frac{3}{h_\alpha}\overset{\alpha}{u}{}_2^{1}$$

$$\overset{\alpha}{\bar{u}}{}_2^{4} = \frac{3}{h_\alpha}\overset{\alpha}{u}{}_2^{1} + \frac{7}{h_\alpha}\overset{\alpha}{u}{}_2^{3}$$

and

$$
\begin{aligned}
&\overset{\alpha}{\bar{u}}_1{}^0 = 0\\
&\overset{\alpha}{\bar{u}}_1{}^2 = \frac{3}{h_\alpha{}^2}\overset{\alpha}{u}_1{}^0\\
&\overset{\alpha}{\bar{u}}_1{}^4 = \frac{10}{h_\alpha{}^2}\overset{\alpha}{u}_1{}^0 + \frac{35}{h_\alpha{}^2}\overset{\alpha}{u}_1{}^2\\
&\overset{\alpha}{\bar{u}}_2{}^1 = 0\\
&\overset{\alpha}{\bar{u}}_2{}^3 = \frac{15}{h_\alpha{}^2}\overset{\alpha}{u}_2{}^1
\end{aligned}
\tag{32}
$$

equations for the face variables: for second order theory

$$
\overset{\alpha}{S}_1{}^+ = 2\overset{\alpha}{u}_1{}^0 + 7\overset{\alpha}{u}_1{}^2 + \frac{1}{5}\overset{\alpha}{A}_1{}^+
$$

$$
\overset{\alpha}{S}_2{}^- = 5\overset{\alpha}{u}_2{}^1 + \frac{1}{3}\overset{\alpha}{A}_2{}^-
$$

for fourth order theory (33)

$$
\overset{\alpha}{S}_1{}^+ = 2\overset{\alpha}{u}_1{}^0 + \frac{60}{7}\overset{\alpha}{u}_1{}^2 + \frac{66}{7}\overset{\alpha}{u}_1{}^4 + \frac{2}{21}\overset{\alpha}{A}_1{}^+
$$

$$
\overset{\alpha}{S}_2{}^- = \frac{28}{5}\overset{\alpha}{u}_2{}^1 + \frac{42}{5}\overset{\alpha}{u}_2{}^3 + \frac{2}{5}\overset{\alpha}{A}_2{}^-,
$$

where

$$
\overset{\alpha}{A}_1{}^+ = \frac{h_\alpha}{2\mu_\alpha}\overset{\alpha}{R}_1{}^- - \frac{h_\alpha}{2}\partial_1\overset{\alpha}{S}_2{}^-
$$

$$
\overset{\alpha}{A}_2{}^- = \frac{h_\alpha}{2(2\mu_\alpha + \lambda_\alpha)}\overset{\alpha}{R}_2{}^+ - \frac{h_\alpha}{2}\cdot\frac{\lambda_\alpha}{(2\mu_\alpha + \lambda_\alpha)}\partial_1\overset{\alpha}{S}_1{}^+
$$

continuity conditions:

$$
\begin{aligned}
&\overset{2}{S}_2{}^- + \overset{1}{S}_2{}^- = 0\\
&\overset{2}{S}_1{}^+ = \overset{1}{S}_1{}^+,\\
&\overset{2}{R}_1{}^- + \overset{1}{R}_1{}^- = 0\\
&\overset{2}{R}_2{}^+ = \overset{1}{R}_2{}^+.
\end{aligned}
\tag{34}
$$

Equations (28)–(34) constitute the governing equations for $P \to x_1$. To obtain the dispersion relation, equations of motion, eqns (28), are written in terms of generalized displacements by using the constitutive equations, eqns (29) and (30). Substituting a trial solution of the form

$$A \exp[i(kn_jx_j - \omega t)] \tag{35}$$

into the resulting equations and requiring that a nontrivial solution exists, the dispersion relation is found. In eqn (35), i is the imaginary number; the n_j are the components of the unit vector defining the direction of propagation (for $P \to x_1$, $\underset{\sim}{n} = (1, 0, 0)$); A denotes the amplitude.

ASSESSMENT OF THE APPROXIMATE THEORY

To appraise the present approximate theory, the dispersion curves predicted by the approximate theory are compared in this section with those derived from the exact theory.

The exact dispersion relations for various waves propagating in a layered composite are already established in the literature and can be found in [2, 5]. For waves propagating normal to layering, the exact analysis indicates that the dispersion curves described on the (ω, k) plane are periodic in k with the period π/Δ and that there are frequency bands, called stopping bands, in which no waves with real-valued wave numbers can propagate. The existence of the stopping

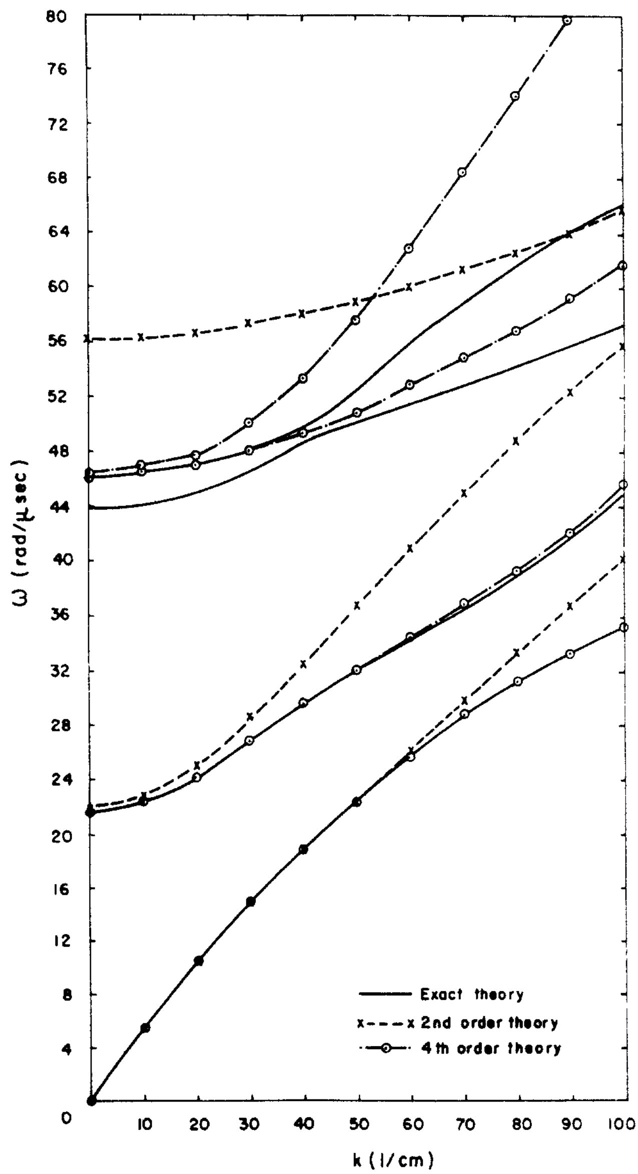

Fig. 3. Approximate and exact spectra for $P \to x_1$ (thornel-carbon phenolic composite).

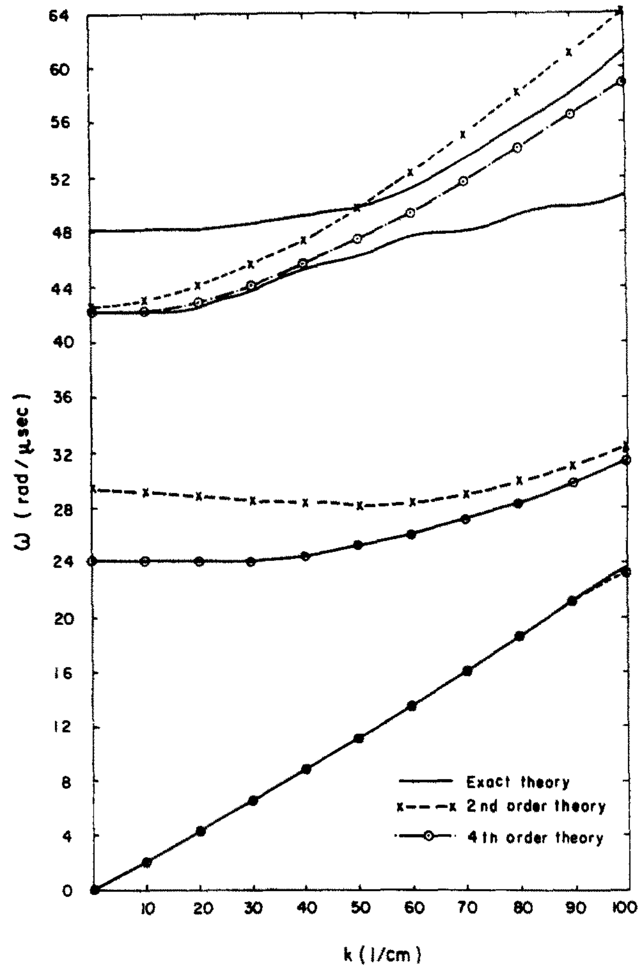

Fig. 4. Approximate and exact spectra for $SV \to x_1$ (thornel-carbon phenolic composite).

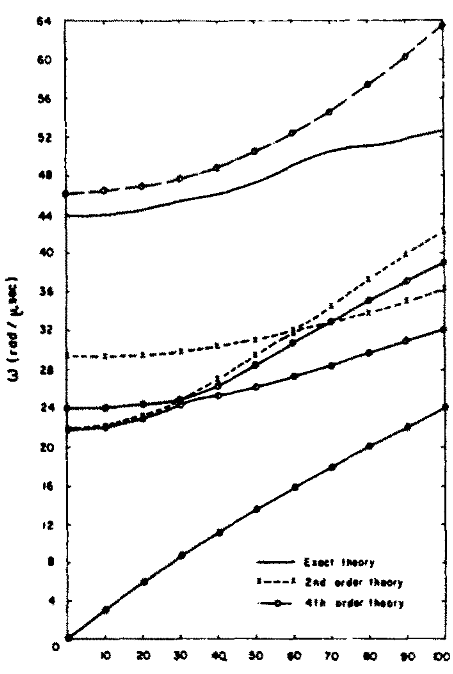

Fig. 5. Approximate and exact spectra for $SH \to x_1$ (thornel-carbon phenolic composite).

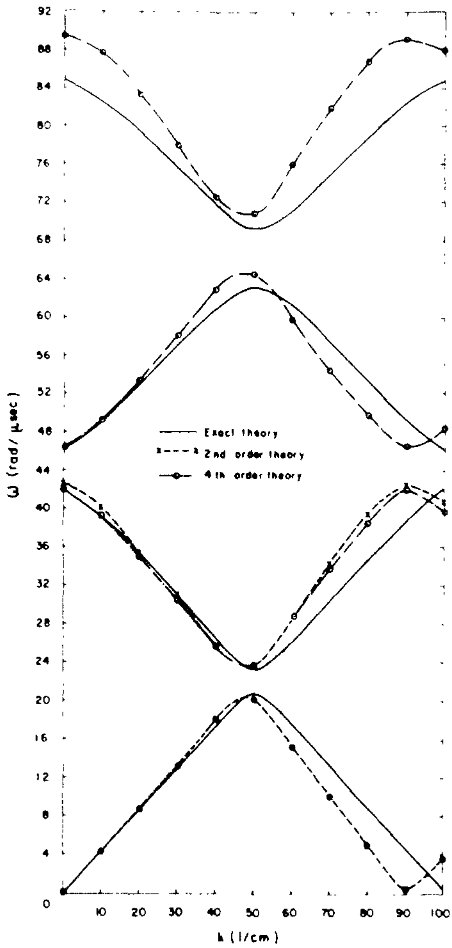

Fig. 6. Approximate and exact spectra for $P \rightarrow x_2$ (thornel-carbon phenolic composite).

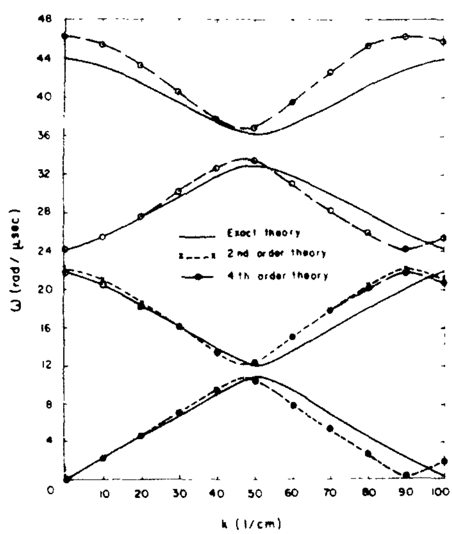

Fig. 7. Approximate and exact spectra for $SV \rightarrow x_2$ (thornel-carbon phenolic composite).

Table 2. Properties of Sun's material

$\frac{h_1}{2h_1}$	$\frac{h_2}{2h_1}$	$n_1 = \frac{h_1}{\Delta}$	$n_2 = \frac{h_2}{\Delta}$	$\frac{\rho_1}{\rho_2}$	$\frac{\rho_2}{\rho_2}$	$\frac{\mu_1}{\mu_2}$	$\frac{\mu_2}{\mu_2}$	$\frac{\lambda_1}{\mu_2}$	$\frac{\lambda_2}{\mu_2}$
0.5	0.125	0.8	0.2	3	1	50	1	75	2.333

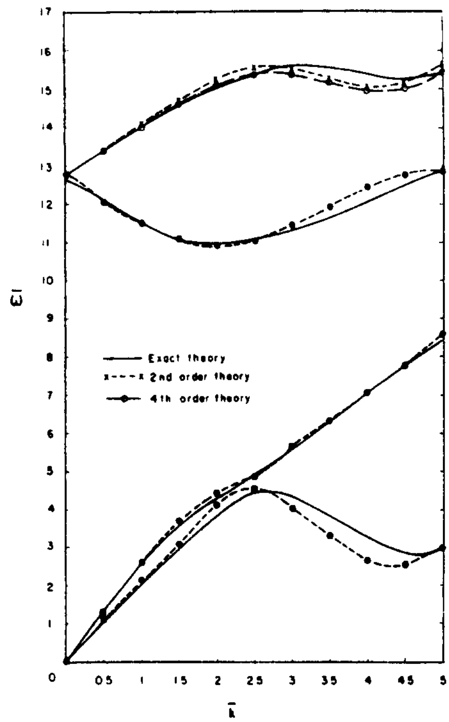

Fig. 8. Approximate and exact spectra for waves propagating on the x_1x_2 plane obliquely with the inclination angle $\alpha = 75°$ from the x_1 axis.

bands for such waves implies that for $P \to x_2$ and $SV \to x_2$ a layered composite acts as a mechanical filter allowing only certain frequencies to pass. This is experimentally verified in [13]. The periodic and banded structure of the spectrum is not present for the other waves propagating parallel and obliquely to the layering.

The approximate dispersion curves are obtained by using second and fourth order theories. In the analysis we choose the distribution functions ϕ_n as Legendre polynomials and accordingly for the second order theory we take the values of γ_k, c'_{nk}, etc. from the table of Part 1 [12]. The exact dispersion curves for $P \to x_1$, $SV \to x_1$, $SH \to x_1$, $P \to x_2$ and $SV \to x_2$ are found numerically by solving the exact frequency equations. The exact dispersion curves for the inclined waves are taken from [14].

The comparison of the approximate and exact dispersion curves of waves propagating in x_1 and x_2 directions is made for thornel-carbon phenolic composites with the constituent properties given in Table 1. This material was used by Whittier *et al.* [15] in their experiments involving the propagation of transient waves. The comparison for inclined waves is presented for Sun's material which was used by Sun *et al.* in their study involving the development of a first order effective stiffness theory for layered composites [3]. The properties of Sun's material are listed in Table 2.

The numerical results are presented in Figs. 3–8. The comparisons are made on the (ω, k) plane rather than (c, k) plane (where $c = \omega/k$ designetes the phase velocity). This preference is made because the use of the (ω, k) plane makes it possible to see the agreement between the dispersion curves in the whole range of k's and all modes of propagation (since unlike the fundamental modes, higher order branches of the spectrum have finite cut-off frequencies but infinite cut-off phase velocities as $k \to 0$). In Fig. 8 the dispersion curves are shown in the $(\bar{\omega}, \bar{k})$ plane where $\bar{w}$ and $\bar{k}$ designate respectively the nondimensional wave number and frequency defined by

$$\bar{k} = 2h_1 k; \qquad \bar{\omega} = \frac{2h_1\omega}{\left(\frac{\mu_2}{\rho_2}\right)^{1/2}}.$$

Some general remarks regarding Figs. 3–8 showing the spectra for waves propagating in various directions are now in order. First it must be observed that the match between the exact and approximate dispersion curves is excellent. The cut-off phase velocities (i.e. phase velocities at $k = 0$) of the fundamental branches predicted by the exact and the approximate theories agree exactly. The match between the approximate and exact dispersion curves and the cut-off frequencies (i.e. frequencies at $k = 0$) improves as the order of the theory increases. To observe the latter point more clearly, Fig. 4 showing the spectrum for $SV \to x_1$ will be referred to. The figure shows that the fundamental branch of the spectrum predicted by the second order theory matches fairly well that of the exact theory. However, this agreement disappears for the second branch. In fact the approximate second branch lies far above the exact and has the cut-off frequency of approximately 29.40 rad/μsec compared to the exact cut-off frequency of 24 rad/μsec. On the other hand, when the fourth order theory is used the approximate second branch comes down to agree quite well with the exact and to have a cut-off frequency of approximately equal to 24 rad/μsec.

Figures 2, 6 and 7 reveal that the approximate theory predicts very well both the banded and periodic structures of the spectra for waves propagating normal to the layering. From these figures it can be concluded that an increase in the orders of the theory and continuity conditions for $P \to x_2$ and $SV \to x_2$ increases respectively the lengths of the intervals $0 \leq k \leq k^*$ and $0 \leq \omega \leq \omega^*$ on which the approximate theory is valid (the approximate theory is said to be valid in a region of the (ω, k) plane if the approximate dispersion curves approximates adequately the exact dispersion curves in that region).

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REFERENCES

1. G. W. Postma, Wave propagation in a stratified medium. *Geophysics* **20**, 780 (1955).
2. S. M. Rytov, Acoustical propagation of a thinly laminated medium. *Soviet Phys. Acoustics* **2**, 68 (1956).
3. C. T. Sun, J. D. Achenbach and G. Herrmann, Continuum theory for a laminated medium. *J. Appl. Mech.* **35**, 467 (1968).
4. D. S. Drumheller and A. Bedford, Wave propagation in elastic laminates using a second order microstructure theory. *Int. J. Solids Structures* **10**, 61 (1974).
5. J. D. Achenbach and G. Herrmann, Wave motion in solids with lamellar structuring. *Dynamics of Structured Solids*, p. 23. The American Society of Mechanical Engineers (1968).
6. G. Herrmann, T. J. Delph and R. K. Kaul, New results on continuum modelling of composites. *Proc. 2nd Int. Symp. on Continuum Models of Descrete Systems*. The University of Waterloo Press.
7. A. Bedford and M. Stern, Toward a diffusing continuum theory of composite materials. *J. Appl. Mech.* **38**, 8 (1971).
8. A. Bedford and M. Stern, A multicontinuum theory for composite elastic materials. *Acta Mechanica* **14**, 85 (1972).
9. H. D. McNiven and Y. Mengi, A mixture theory for elastic laminated composites. *Int. J. Solids Structures* **15**, 281 (1979).
10. G. A. Hegemier, On a theory of interacting continua for wave propagation in composites. *Dynamics of composite materials*, p. 70. The American Society of Mechanical Engineers (1972).
11. G. A. Hegemier and T. C. Bache, A general continuum theory with microstructure for wave propagation in elastic laminated composites. *J. Appl. Mech.* **41**, 101 (1974).
12. Y. Mengi, A new approach for developing dynamic theories for structural elements, Part 1: Application to thermoelastic plates. *Int. J. Solids Structures* **16**, 1157–1170 (1980).
13. C. W. Robinson, *Shear waves in layered composites. Rep. No. SCL-RR*-720351. Sandia Laboratories, Livermore, California (1972).
14. C. Sve, Time harmonic waves travelling obliquely in a periodically laminated medium. *J. Appl. Mech.* **38**, 477 (1971).
15. J. S. Whittier and J. C. Peck, Experiments on dispersive pulse propagation in laminated composites and comparison with theory. *J. Appl. Mech.* **36**, 485 (1969).